L-FUNCTIONS ASSOCIATED TO MODULAR FORMS

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ABSTRACT. Modular forms are classical entities found across various mathematical domains like number theory, representation theory, and mathematical physics, have gained prominence for their pivotal role in mathematics. Notably, they played a crucial part in proving Fermat's Last Theorem by affirming the Shimura-Taniyama-Weil conjecture, linking modular forms to elliptic curves. Moreover, a significant link between modular forms and arithmetic lies in L-functions, with the Riemann ζ-function serving as a fundamental example. This exposition focuses on studying L-functions associated with modular forms. Beginning with the theory of Hecke operators as outlined in §$3.1 - 3.5$ of [4], we progress to analyzing the convergence criteria for these L-functions. Subsequently, drawing from §3.6 of [4], we establish an Euler product representation using Hecke operators. Our ultimate goal is to develop analytic continuations and functional equations for these L-functions.

CONTENTS

1. DOUBLE COSET OPERATORS

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and k be an integer, then we denote the space of modular forms of weight k with respect to Γ by $\mathcal{M}_k(\Gamma)$, and the cusp forms by $\mathcal{S}_k(\Gamma)$.

For any congruence subgroups Γ_1 and Γ_2 of $SL_2(\mathbb{Z})$, a set of double coset operators maps $\mathcal{M}_k(\Gamma_1)$ to $\mathcal{M}_k(\Gamma_2)$, preserving cusp forms. These operators are linear.

Example 1.1. In the case where $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$, specific double coset operators $\langle n \rangle$ and T_n for all $n \in \mathbb{Z}^+$ are the Hecke operators, commuting endomorphisms of the vector space $\mathcal{M}_k(\Gamma_1(N))$ and the subspace $\mathcal{S}_k(\Gamma_1(N))$.

Let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, then the set $\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 : \gamma_i \in \Gamma_i, i = 1, 2\}$ is a double coset in $\mathrm{GL}_2^+(\mathbb{Q})$. The group Γ_1 acts on the double coset $\Gamma_1\alpha\Gamma_2$ by left multiplication, partitioning it into orbits. An orbit it $\Gamma_1\beta$ with $\beta = \gamma_1\alpha\gamma_2$, and the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ is a disjoint union $\sqcup\Gamma_1\beta_j$ for some choice of orbit representatives β_j. In fact, this union is finite as shown by next two lemmas.

Lemma 1. *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and let α be an element of $GL_2^+(\mathbb{Q})$. Then $\alpha^{-1}\Gamma\alpha \cap SL_2(\mathbb{Z})$ is also a congruence subgroup of $SL_2(\mathbb{Z})$.*

Proof. Lemma 5.1.1. in [1]. ■

Lemma 2. *Let Γ_1 and Γ_2 be congruence subgroups of $SL_2(\mathbb{Z})$, and let α be an element of $GL_2^+(\mathbb{Q})$. Define $\Gamma_3 = \alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$, a subgroup of Γ_2. Then left multiplication by α,*

$$\Gamma_2 \to \Gamma_1\alpha\Gamma_2, \quad \gamma_2 \mapsto \alpha\gamma_2,$$

induces a natural bijection from the coset space $\Gamma_3\backslash\Gamma_2$ to the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. In simpler terms, the set $\{\gamma_{2,j}\}$ is a set of coset representatives for $\Gamma_3\backslash\Gamma_2$ if and only if the set $\{\beta_j\} = \{\alpha\gamma_{2,j}\}$ is a set of orbit representatives for $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$.

Proof. Lemma 5.1.2. in [1]. ■

We know that any two congruence subgroups Γ_1 and Γ_2 of $\mathrm{SL}_2(\mathbb{Z})$ are commensurable: the indices $[\Gamma_1 : \Gamma_1 \cap \Gamma_2]$ and $[\Gamma_2 : \Gamma_1 \cap \Gamma_2]$ are finite. Since, $\alpha^{-1}\Gamma\alpha \cap SL_2(\mathbb{Z})$ is a congrunce subgroup, the coset space $\Gamma_3\backslash\Gamma_2$ is finite and hence so is the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$.

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $k \in \mathbb{Z}$, the weight-k α operator on functions $f : \mathbb{H} \to \mathbb{C}$ is given by

$$(f[\alpha]_k)(z) = (\det \alpha)^{k-1} j(\alpha, z)^{-k} f(\alpha(z)), \quad z \in \mathbb{H}.$$

where

$$j(\alpha, z) = cz + d, \quad \alpha(z) = \frac{az + b}{cz + d}.$$

Definition 1. For congruence subgroups Γ_1 and Γ_2 of $SL_2(\mathbb{Z})$ and $\alpha \in GL_2^+(\mathbb{Q})$, the weight-$k$ $\Gamma_1\alpha\Gamma_2$ operator maps functions $f \in \mathcal{M}_k(\Gamma_1)$ to

$$f[\Gamma_1\alpha\Gamma_2]_k = (\det\alpha)^{k/2-1}\sum_j f[\beta_j]_k$$

where $\{\beta_j\}$ are orbit representatives, meaning $\Gamma_1\alpha\Gamma_2 = \bigcup_j \Gamma_1\beta_j$ is a disjoint union.

Proposition 3. *The double coset operator is well defined: it does not depend on the choice of the orbit reprsentatives β_j. The **weight-k** $\Gamma_1\alpha\Gamma_2$ operator takes $f \in \mathcal{M}_k(\Gamma_1)$ to $f[\Gamma_1\alpha\Gamma_2]_k \in \mathcal{M}_k(\Gamma_2)$. In particular, if f is a cup form then $f[\Gamma_1\alpha\Gamma_2]_k$ is also a cusp form.*

Proof. We need to show that $f[\Gamma_1\alpha\Gamma_2]_k$ is Γ_2-invariant and is holomorphic at the cusps. To demonstrate invariance, observe that any $\gamma_2 \in \Gamma_2$ permutes the orbit space $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ by right multiplication. In other words, the map $\gamma_2 : \Gamma_1\backslash\Gamma_1\alpha\Gamma_2 \to \Gamma_1\backslash\Gamma_1\alpha\Gamma_2$ defined by $\Gamma_1\beta \mapsto \Gamma_1\beta\gamma_2$ is well-defined and bijective. Thus, if $\{\beta_j\}$ is a set of orbit representatives for $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$, then $\{\beta_j\gamma_2\}$ is also a set of orbit representatives. Thus

$$(f[\Gamma_1\alpha\Gamma_2]_k)[\gamma_2]_k = (\det\alpha)^{k/2-1}\sum_j f[\beta_j\gamma_2]_k = f[\Gamma_1\alpha\Gamma_2]_k.$$

∎

Special cases of the double coset operator $[\Gamma_1\alpha\Gamma_2]_k$ arise when:

(1) Γ_1 contains Γ_2. Taking $\alpha = I$ makes the double coset operator $f[\Gamma_1\alpha\Gamma_2]_k = f$, representing the natural inclusion of the subspace $M_k(\Gamma_1)$ in $M_k(\Gamma_2)$, an injection.

(2) $\alpha^{-1}\Gamma_1\alpha = \Gamma_2$. Here, the double coset operator is $f[\Gamma_1\alpha\Gamma_2]_k = f[\alpha]_k$, representing the natural translation from $M_k(\Gamma_1)$ to $M_k(\Gamma_2)$, an isomorphism.

(3) Γ_1 is a subset of Γ_2. Taking $\alpha = I$ and letting $\{\gamma_{2,j}\}$ be a set of coset representatives for $\Gamma_1\backslash\Gamma_2$ makes the double coset operator $f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\gamma_{2,j}]_k$, the natural trace map that projects $M_k(\Gamma_1)$ onto its subspace $M_k(\Gamma_2)$ by symmetrizing over the quotient, a surjection.

2. The $\langle n\rangle$ and T_n operators

Fix $N \in \mathbb{N}$. Let S^+ be an additive subgroup of $\mathbb{Z}$, i.e., $S^+ = M\mathbb{Z}$ for some integer M. Let $S^\times$ be a multiplicative subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. We shall also use $S^\times$ to indicate the pre-image of $S^\times$ under the natural map $\mathbb{Z} \to S^\times$. Let

$$\begin{aligned} X_n &= X_n(N, S^\times, S^+) \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{Z}) : ad - bc = n, N|c, a \in S, b \in S^+ \right\}. \end{aligned}$$

Example 2.1. The standard Hecke congruence subgroups can all be written in this form:

$$\begin{aligned} \Gamma_1(N) &= X_1(N, 1, \mathbb{Z}) \\ \Gamma(N) &= X_1(N, 1, N\mathbb{Z}) \\ \Gamma_0(N) &= X_1(N, (\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{Z}). \end{aligned}$$

Lemma 4. *$X_1(N, S^\times, S^+)$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. Clearly if $S^+ = M\mathbb{Z}$, then

$$\Gamma(MN) \subset X_1(N, 1, S^+) \subset X_1(N, S^\times, S^+).$$

■

Lemma 5. *The set $X_n = X_n(N, S^\times, S^+)$ is invariant under left and right action of X_1. Furthermore, we have*

$$| X_1(N, S^\times, S^+) \backslash X_n(N, S^\times, S^+) | < \infty.$$

Proof. See page 9 in [2]. ■

Thus there is a finite set of orbit representatives $\alpha_i \in X_n$ such that

$$X_n = \bigsqcup_i X_1 \alpha_i.$$

Moreover, multiplying the orbits on the right by any fixed $\gamma \in X_1$ simply permutes the orbits: $\bigsqcup_i X_1 \alpha_i \gamma = \bigsqcup_i X_1 \alpha_i$.

Definition 2. Define the n-th Hecke operator on $M_k(X_1)$ as the map which sends $f \in M_k(X_1)$ to the finite sum

$$T_n(f) := n^{\frac{k}{2}-1} \sum_i f[\alpha_i]_k. \tag{1}$$

Remark. The way we have defined the operators T_n is taken from [2] and is different than the defintion given by Shimura (page 70 of [4]), however both are equivalent.

Theorem 6. *Consider the group $\Gamma = X_1(N, S^\times, S^+)$ as described above. For any $k, n \geq 1$, the Hecke operator defined in equation 1 is a linear map from $\mathcal{M}_k(\Gamma)$ to $\mathcal{M}_k(\Gamma)$ and from $\mathcal{S}_k(\Gamma)$ to $\mathcal{S}_k(\Gamma)$.*

Proof. Suppose $f \in \mathcal{M}_k(\Gamma)$ and $\gamma \in \Gamma$. Then for any α_i, since $f|\gamma\alpha_i = f|\alpha_i$, we see that the map does not depend on the choice of orbit representatives and is thus well-defined. Moreover,

$$\begin{aligned} T_n(f)[\gamma]_k &= \left(n^{k/2-1} \sum_i f[\alpha_i]_k \right) [\gamma_i]_k \\ &= n^{k/2-1} \sum_i f[\alpha_i \gamma]_k \\ &= T_n(f) \end{aligned}$$

where the final equality follows from the fact that γ simply permutes the orbits. We know that for any $\alpha \in \mathrm{GL}(2,\mathbb{Q})^+$, $f[\alpha]_k$ is holomorphic at every cusp, and hence $T_n(f)$ is also holomorphic at every cusp. Thus, $T_n(f) \in \mathcal{M}_k(\Gamma)$, furthermore if $f \in \mathcal{S}_k(\Gamma)$, then the constant coefficient in the Fourier expansion of $f[\alpha]_k$ is 0. Therefore, $T_n(f) \in \mathcal{S}_k(\Gamma)$. ■

We can give an explicit description in the case of $\Gamma = \Gamma_1(N) = X_1(N, 1, \mathbb{Z})$. From now onwards, for each $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, we fix $\sigma_a \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\sigma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \pmod{N},$$

where a^{-1} denotes the multiplicative inverse of $a \pmod{N}$.

Theorem 7. *We have the following decompostion of $X_n = X_n(N, 1, \mathbb{Z})$*

$$X_n = \bigsqcup_{\substack{ad=n \\ a>0 \\ (a,N)=1}} \bigsqcup_{b=0}^{d-1} \Gamma_1(N)\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where the first disjoint union is taken over all positive integers a dividing n that are coprime to N.

Proof. See Theorem 8.1.7. in [2]. ■

Corollary 7.1. *For a prime p*

$$X_p(N, 1, \mathbb{Z}) = \bigsqcup_{b=0}^{p-1} \Gamma_1(N) \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \quad (p \mid N),$$

and

$$X_p(N, 1, \mathbb{Z}) = \bigsqcup_{b=0}^{p-1} \Gamma_1(N) \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \bigsqcup \Gamma_1(N) \begin{pmatrix} m & n \\ N & p \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \quad (p \nmid N).$$

Corollary 7.2. *For $(n, N) = 1$, we have an explicit description of T_n*

$$T_n(f) = n^{\frac{k}{2}-1} \sum_{\substack{ad=n \\ a>0}} \sum_{b=0}^{d-1} f\left[\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right]_k$$

where $\begin{pmatrix} m & n \\ N & p \end{pmatrix} \in SL_2(\mathbb{Z})$

Remark. It is easy to see that for a prime p we have the following equality.

$$X_p(N, 1, \mathbb{Z}) = \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N).$$

Thus the definition of Hecke operators given in [1] coincides with the one given here.

Now we define another type of Hecke operator called the **diamond** operator. The mapping $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^*$, which sends a/b to $d \pmod N$, is a surjective homomorphism with kernel $\Gamma_1(N)$. This demonstrates that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and induces an isomorphism

$$\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^*, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod N.$$

Take any $\alpha \in \Gamma_0(N)$, set $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$, and consider the weight k double coset operator $[\Gamma_1 \alpha \Gamma_2]_k$. Since, $\Gamma_1(N) \lhd \Gamma_0(N)$, this operator is the special case (2) from §1, taking a modular form $f \in \mathcal{M}_k(\Gamma_1(N))$ to

$$f[\Gamma_1(N)\alpha\Gamma_1(N)]_k = f[\alpha]_k, \quad \alpha \in \Gamma_0(N),$$

again in $\mathcal{M}_k(\Gamma_1(N))$. Hence, the group $\Gamma_0(N)$ acts on $\mathcal{M}_k(\Gamma_1(N))$, and as its subgroup $\Gamma_1(N)$ acts trivially, this constitutes an action of the quotient $(\mathbb{Z}/N\mathbb{Z})^*$. The action of $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, determined by $d \pmod N$ and denoted $\langle d \rangle$, is

$$\langle d \rangle : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$$

gievn by $\langle d \rangle f = f[\alpha]_k$ for any $\alpha = \left(\begin{smallmatrix} a & b \\ c & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ with $\delta \equiv d \pmod N$. We observe that σ_d where d varies in $(\mathbb{Z}/N\mathbb{Z})^*$ is a set of representatives of Γ_0/Γ_1. Thus we can give the following definition:

Definition 3. For each $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, define the **diamond** operator $\langle d \rangle$ on $\mathcal{M}_k(\Gamma_1(N))$ as $\langle d \rangle f = f[\sigma_d]_k$.

2.1. **Nebentypus.** For any Dirichlet character χ modulo N, we define the following vector subspace of $M_k(\Gamma_1(N))$:

$$\mathcal{M}_k(N, \chi) := \{f \in M_k(\Gamma_1(N)) : f[\gamma]_k = \chi(d)f, \text{ for all } \gamma \in \Gamma_0(N)\}.$$

Where, d is the lower right entry of $\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In particular, if χ is the trivial character, then $\mathcal{M}_k(N,\chi) = M_k(\Gamma_0(N))$. It is also clear that we have the following equality

$$\mathcal{M}_k(N,\chi) := \{f \in M_k(\Gamma_1(N)) : \langle d \rangle f = \chi(d) f, \text{ for all } d \in (\mathbb{Z}/N\mathbb{Z})^\times\}.$$

Theorem 8. *We have the following decompositions*

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_\chi \mathcal{M}_k(N,\chi),$$
$$\mathcal{S}_k(\Gamma_1(N)) = \bigoplus_\chi \mathcal{S}_k(N,\chi),$$

where the direct sum is over all Dirichlet characters modulo N.

Proof. This is a standard result from representation theory of finite groups. However, we present an elementary argument. We first show that every $f \in \mathcal{M}_k(\Gamma_1(N))$ can be written as a sum of functions $f_\chi \in \mathcal{M}_k(N,\chi)$. Let

$$f_\chi = \frac{1}{\phi(N)} \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^*} \overline{\chi}(d) f[\sigma_d]_k,$$

where $\sigma_d \in \Gamma_0(N)$ is as above. This well-defined since $f \in \mathcal{M}_k(\Gamma_1(N))$. Indeed, $f_\chi \in \mathcal{M}_k(N,\chi)$. For any $a \in (\mathbb{Z}/N\mathbb{Z})^*$, we have

$$\begin{aligned} f_\chi[\sigma_a]_k &= \frac{1}{\phi(N)} \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^*} \overline{\chi}(d) f[\sigma_d]_k[\sigma_a]_k \\ &= \frac{1}{\phi(N)} \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^*} \overline{\chi}(d) f[\sigma_d \sigma_a]_k \\ &= \frac{1}{\phi(N)} \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^*} \overline{\chi}(da)\chi(a) f[\sigma_d \sigma_a]_k. \end{aligned}$$

Now as d runs through coprime residue classes modulo N, so does da. We deduce that

$$f_\chi[\sigma_a]_k = \chi(a) f_\chi.$$

Moreover, f_χ is holomorphic at all cusps because $f[\sigma_d]_k$ is. Therefore, $f_\chi \in \mathcal{M}_k(N,\chi)$. Now

$$\sum_\chi f_\chi = \sum_{d \in (\mathbb{Z}/N\mathbb{Z})^*} \left(\frac{1}{\phi(N)} \sum_\chi \overline{\chi}(d) \right),$$

where the inner sum is 0, unless $d = 1$, in which case it is 1, so that

$$f = \sum_\chi f_\chi.$$

We now show that the sum is direct, let $g \in \mathcal{M}_k(N, \psi)$. For any character χ, we compute

$$g_\chi = \left(\frac{1}{\phi(N)} \sum_\chi \psi\overline{\chi}(d) \right) g = \begin{cases} g & \text{if } \chi = \psi \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if

$$g \in \mathcal{M}_k(N, \psi) \cap \bigoplus_{\chi \neq \psi} \mathcal{M}_k(N, \chi),$$

then

$$g = g_\psi = \left(\sum_{\chi \neq \psi} g_\chi \right)_\psi = 0.$$

This proves the first decomposition. This argument also implies the decomposition for cusp forms, because the constant coefficient of $f[\sigma_d]_k$ is zero if that of f. ■

Remark. Notice that $\mathcal{M}_k(N, \chi) = 0$ if χ has a different parity from k, i.e., if $\chi(-1) \neq (-1)^k$. This follows by taking $\gamma = -I$ in the definition and recalling that $f[-I]_k = (-1)^k f$.

The spaces $\mathcal{M}_k(N, \chi)$ include many of the most important examples of modular forms, and will be our basic object of study in what follow. Let T_n be the n-th Hecke operator of $\mathcal{M}_k(X_1(N, 1, \mathbb{Z})) = \mathcal{M}_k(\Gamma_1(N))$, we know that it is of the form

$$T_n(f) = n^{\frac{k}{2}-1} \sum_{\substack{ad=n \\ a>0 \\ (a,N)=1}} \sum_{b=0}^{d-1} f \left[\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right]_k,$$

where $f \in \mathcal{M}_k(\Gamma_1(N))$. Suppose that $f \in \mathcal{M}_k(N, \chi)$, then we have $f[\sigma_a]_k = \chi(a) f$ si that

$$T_n(f) = n^{\frac{k}{2}-1} \sum_{\substack{ad=n \\ a>0}} \chi(a) \sum_{b=0}^{d-1} f \left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right]_k.$$

Notice that we dropped the condition that $(a, N) = 1$ since $\chi(a) = 0$ if $(a, N) \neq 1$. We adopt the following notation: for any $f \in \mathcal{M}_k(\Gamma_1(N))$, we write its fourier expansion as

$$f(z) = \sum_{m=0}^{\infty} a_m(f) q^m.$$

Remark. Note that the operators $\langle d \rangle$ and T_n preserve the decomposition $M_k(\Gamma_1(N)) = \bigoplus_\chi \mathcal{M}_k(N, \chi)$. Clearly, if $f \in \mathcal{M}_k(N, \chi)$, then by definition

$\langle d\rangle f \in \mathcal{M}_k(N,\chi)$. Furthermore, since $(\langle d\rangle T_n)f = (T_n\langle d\rangle)f$, we see that $T_n(f) \in \mathcal{M}_k(N,\chi)$

Theorem 9. *Let $f \in \mathcal{M}_k(N,\chi)$. Then*

$$a_m(T_n(f)) = \sum_{d|(m,n)} \chi(d)d^{k-1}a_{\frac{mn}{d^2}}(f). \tag{2}$$

Proof. We have

$$T_n(f) = n^{\frac{k}{2}-1}\sum_{\substack{ad=n\\a>0}}\chi(a)\sum_{b=0}^{d-1} f\left[\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\right]_k.$$

Now

$$\sum_{b=0}^{d-1} f\left[\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\right]_k = \frac{1}{n}\sum_{ad=n,a>0}\Big(\frac{n}{d}\Big)^k\chi(a)\sum_{b=0}^{d-1} f\Big(\frac{az+b}{d}\Big).$$

We evaluate

$$\begin{aligned}\sum_{b=0}^{d-1} f\Big(\frac{az+b}{d}\Big) &= \sum_{b=0}^{d-1}\sum_{m=0}^{\infty} a_m(f)e^{\frac{2\pi m(az+b)}{d}}\\ &= \sum_{m=0}^{\infty}\left(\sum_{b=0}^{d-1} e^{\frac{2\pi mb}{d}}\right)a_m(f)e^{\frac{2\pi maz}{d}}\\ &= d\sum_{r=0}^{\infty} a_{dr}(f)e^{2\pi arz}\end{aligned}$$

the last equality follows since $\sum_{b=0}^{d-1} e^{\frac{2\pi mb}{d}} = d$ if $d \mid m$ and is 0 otherwise. Therefore,

$$\begin{aligned}(T_nf)(z) &= \sum_{\substack{ad=n\\a>0}}\Big(\frac{n}{d}\Big)^{k-1}\chi(a)\sum_{r=0}^{\infty} a_{dr}(f)e^{2\pi arz}\\ &= \sum_{m=0}^{\infty}\left(\sum_{\substack{ad=n\\a>0\\ar=m}}\Big(\frac{n}{d}\Big)^{k-1}\chi(a)a_{dr}(f)\right)e^{2\pi mz}.\end{aligned}$$

This implies that

$$a_m(T_nf) = \sum_{d|(m,n)}\chi(d)d^{k-1}a_{\frac{mn}{d^2}}(f).$$

■

The following identity is at the heart of all Euler factorizations of "nice" modular forms.

Proposition 10. *The n-th Hecke operators when restricted to $\mathcal{M}_k(N,\chi)$ satisfy the following:*

$$T_mT_n = \sum_{d|(n,m)} d^{k-1}\chi(d)T_{\frac{mn}{d^2}}. \tag{3}$$

In particular, when n is a prime power we have

$$T_{p^r} = T_pT_{p^{r-1}} - p^{k-1}\chi(p)T_{p^{r-2}}, \quad r \geq 2.$$

Proof. By comparing the fourier coefficients of both sides and using the "combinatorial" lemma. ∎

Corollary 10.1. *If $(m,n) = 1$, then $T_mT_n = T_nT_m = T_{mn}$. Furthermore, formally we have the following Euler product*

$$\sum_{n=1}^{\infty} T_n n^{-s} = \prod_{p|N}(1 - T_pp^{-s})^{-1} \cdot \prod_{p\nmid N}(1 - T_pp^{-s} + \chi(p)p^{k-1-2s})^{-1}. \tag{4}$$

Proof. The first assertion is clear from Proposition 10. Thus, formally we have

$$\sum_{n=1}^{\infty} T_n n^{-s} = \prod_p \left(\sum_{r=0}^{\infty} T_{p^r}p^{-rs}\right).$$

We claim that

$$\left(\sum_{r=0}^{\infty} T_{p^r}p^{-rs}\right) \cdot (1 - T_pp^{-s} + \chi(p)p^{k-1-2s}) = 1,$$

the coefficient of p^{-s} is clearly 0 in the above product. Furthermore, the coefficient of p^{-rs} for $r > 1$ is

$$T_{p^r} - T_{p^{r-1}}T_p + T_{p^{r-2}}\chi(p)p^{k-1} = 0,$$

from Proposition 10, which proves the claim. ∎

Definition 4. A nonzero modular form $f \in \mathcal{M}_k(\Gamma_1(N))$ that is a common eigen-function for the Hecke operators T_n for all $n \in \mathbb{Z}^+$ is a **Hecke eigenform** or simply an **eigenform**. The eigenform $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n$ is normalized when $a_1(f) = 1$.

Proposition 11. *Let $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in \mathcal{M}_k(N,\chi)$ be an eigenform: $T_nf = \lambda_nf$ for $\lambda_n \in \mathbb{C}\setminus\{0\}$, then $a_1(f) \neq 0$ and $a_{mn}(f)a_1(f) = a_m(f)a_n(f)$ for $(m,n) = 1$.*

Proof. Formula 3 says

$$\lambda_n a_1(f) = a_1(T_n f) = a_n(f), \quad \forall n \in \mathbb{Z}^+.$$

Thus if $a_1(f) = 0$, then $a_n(f) = 0$ for all n so $f = 0$, which is not possible. Further, for $(m, n) = 1$

$$a_{mn}(f)a_1(f) = a_1(T_{mn}f)a_1(f) = \lambda_m\lambda_n a_1(f)^2 = a_m(f)a_n(f).$$

■

Proposition 12. *Let* $f(z) = \sum_{n=0}^{\infty} a_n(f)q^n \in \mathcal{M}_k(N, \chi)$ *be an eigenform:* $T_n(f) = \lambda_n f$, *for* $\lambda_n \in \mathbb{C} \setminus \{0\}$. *Then,*

$$\sum_{n=1}^{\infty} \lambda_n n^{-s} = \prod_p (1 - \lambda_p p^{-s} + \chi(p)p^{k-1-2s})^{-1} \quad \textit{(formally).} \tag{5}$$

Conversely, if one has formally

$$\sum_{n=1}^{\infty} a_n(f)n^{-s} = \prod_p (1 - a_p(f)p^{-s} + \chi(p)p^{k-1-2s})^{-1}, \tag{6}$$

then $T_n(f) = a_n(f)f$ *for all* n.

Proof. For $(m, n) = 1$ we have $\lambda_{mn}f = T_{mn}f = T_m T_n f = \lambda_m\lambda_n f$. Thus, $\lambda_{mn} = \lambda_m\lambda_n$. It is also clear from formula 3 that

$$\lambda_{p^r} = \lambda_p\lambda_{p^{r-1}} - p^{k-1}\chi(p)\lambda_{p^{r-2}}, \quad r \geq 2,$$

then the same argument as in Corollary 10.1 implies the first assertion. Furthermore, if f is normaized eigenform: $a_1(f) = 1$, then $\lambda_n = a_n(f)$.

Conversely, the Euler product 6 implies

(1) $a_m n(f) = a_m(f)a_n(f)$ when $(m, n) = 1$,

(2) $a_{p^r}(f) = a_p(f)a_{p^{r-1}}(f) - \chi(p)p^{k-1}a^{p^{r-2}}(f)$ for all primes p and $r \geq 2$.

We note that, for f to be an eigenfoem it need to satisfy $a_m(T_p f) = a_p(f)a_m(f)$ for all prime p and $m \in \mathbb{Z}^+$. If $p \nmid m$ then formula 2 gives $a_m(T_p f) = a_{pm}(f) = a_p(f)a_m(f)$. On the other hand, if $p \mid m$ write $m = p^r m'$, with $r \geq 1$ and $p \nmid m'$. Then

$$\begin{aligned} a_m(T_p f) &= a_{p^{r+1}m'}(f) + \chi(p)p^{k-1}a_{p^{r-1}m'}(f) && \text{formula 2} \\ &= (a_{p^{r+1}}(f) + \chi(p)p^{k-1}a_{p^{r-1}}(f))a_{m'}(f) && \text{condition (2)} \\ &= a_p(f)a_{p^r}(f)a_{m'}(f) && \text{condition (1)} \\ &= a_p(f)a_m(f) && \text{condition (2).} \end{aligned}$$

■

Remark. For a normalized eigenform $f \in \mathcal{S}_k(N, \chi)$, it is clear from the argument in the above proof that the Dirichlet series associated to f has an Euler product:

$$\sum_{n=1}^{\infty} a_n(f) n^{-s} = \prod_p (1 - a_p(f)p^{-s} + \chi(p)p^{k-1-2s})^{-1}.$$

We deal with the convergence properties in the next section.

2.2. **Ramanujan τ-function.** In his fundamental paper of 1916, Ramanujan introduced the τ-function as being the coefficients in the power series expansion of the infinite product

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

He made three conjectures regarding $\tau(n)$:

(1) $\tau(mn) = \tau(m)\tau(n)$ for $(m, n) = 1$,

(2) if p is prime, then $\tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - p^{11}\tau(p^{\alpha-1})$ for $\alpha \geq 1$;

(3) if p is prime, then $|\tau(p)| \leq 2p^{11/2}$.

It is well known that $q \prod_{n=1}^{\infty}(1 - q^n)^{24}$ is a cusp form of weight 12 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ for $q = e^{2\pi i z}$ and $z \in \mathbb{H}$, denoted by Δ. Since $\mathcal{S}_{12} = \mathcal{S}_{12}(\Gamma_1(1))$ is of dimension 1, $\Delta \in \mathcal{S}_{12}$ is automatically an eigenform. It is also clear that Δ is normalised. Thus

$$\sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}.$$

Since, Δ is normalised $T_n(\Delta) = \tau(n)\Delta$. Thus the first two conjectures of Ramanujan follows from 10. The third conjecture of Ramanujan was proved by Deligne using extensive tools from algebraic geometry. He proved more generally that if $f \in \mathcal{S}_k(N, \chi)$ is primitive, then $|a_p(f)| \leq 2p^{k-1}$ (a particular but highly important case of the famous Ramanujan-Petersson conjecture). We cannot say anything about the proof in this text, except that it relies heavily on very difficult algebraic geometry.

2.3. **Basis of cusp forms.** We shall now focus on $\mathcal{S}_k(\Gamma_1)$, and introduce an inner product in the space $\mathcal{S}_k(\Gamma_1)$. Define the hyperbolic measure on the upper half plane,

$$d\mu(z) = \frac{dxdy}{y^2}, \quad z = x + iy \in \mathbb{H}.$$

Let $f, g \in \mathcal{S}_k(\Gamma_1)$, we define

$$\langle f, g \rangle = \int_{\Gamma_1(N)\backslash\mathbb{H}} f(z)\overline{g(z)} \cdot y^{k-2} dxdy, \quad z = x + iy \in \mathbb{H}.$$

Here $f(z)\overline{g(z)} \cdot y^k$ and $y^{-2}dxdy$ are invariant under $\Gamma_1(N)$, therefore, the integral is well-defined if it converges. It also converges (page 74 in [4]). The inner product $\langle f, g\rangle$ is of course hermitian and positive definite; it is called the *Petersson inner product* on $\mathcal{S}_k(\Gamma_1(N))$. The following proposition determines the adjoint of the Hecke operators with respect to the inner product.

Proposition 13. *Let $f, g \in \mathcal{S}_k(\Gamma_1(N))$, and $(n, N) = 1$. Then*

(1) $\langle\langle n\rangle f, g\rangle = \langle f, \langle n^{-1}\rangle g\rangle$,

(2) $\langle T_n(f), g\rangle = \langle f, \langle n^{-1}\rangle T_n(g)\rangle$.

In particular, if $f, g \in \mathcal{S}_k(N, \chi)$, then $\langle T_n(f), g\rangle = \chi(n)\langle f, T_n(g)\rangle$.

Proof. Proposition 5.5.2 in [1]. ■

Proposition 14 (Commutativity of Hecke operators)**.** *Let $d, e \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $n, m \in \mathbb{Z}^+$ then*

(1) $\langle d\rangle T_n = T_n\langle d\rangle$,

(2) $\langle d\rangle\langle e\rangle = \langle e\rangle\langle d\rangle = \langle de\rangle$,

(3) $T_nT_m = T_mT_n$.

Proof. We have already seen (1). Since $\langle d\rangle$ and T_n preserves the decomposition $M_k(\Gamma_1(N)) = \bigoplus_\chi \mathcal{M}_k(N, \chi)$, so it suffices to check (2) and (3) on any arbitrary $f \in \mathcal{M}_k(N, \chi)$. It is immediate to see that (2) holds, and (3) holds because of Proposition 10. ■

Definition 5. A normal operator on a complex inner product space V is a linear operator $T : V \to V$ that commutes with its Hermitian adjoint T^*, that is: $TT^* = T^*T$.

As a consequence of Proposition 13 and 14 we get the following:

Corollary 14.1. *The Hecke operators $\langle n\rangle$ and T_n for n relatively prime to N are normal.*

Theorem 15 (Spectral Theorem)**.** *If there is a set of normal operators on a finite-dimensional inner product space that commute with each other, then the space possesses a set of orthogonal basis vectors that are eigenvectors for all the operators simultaneously.*

In our context of modular forms, we refer to these eigenvectors as eigenforms. This leads to the following conclusion:

Corollary 15.1. *The space $S_k(\Gamma_1(N))$ has an orthogonal basis of simultaneous eigenforms for the Hecke operators $\{\langle n\rangle, T_n : (n, N) = 1\}$.*

Proof. Proposition 14, Corollary 14.1, and Theorem 15 imply the assertion. ■

3. L-functions associated to Modular forms

We begin this section with two rudimentary lemmas about the cusp-forms of level $\Gamma_1(N)$.

Lemma 16. *Suppose $f \in \mathcal{S}_k(T_1(N))$, then*

$$\left|f(x+iy)\right| \leq My^{-k/2}$$

where M is a constant independent of x. Conversely, if $f \in \mathcal{M}_k(T_1(N))$ and $\left|f(x+iy)\right| \leq My^{-k/2}$ with a constant M independent of x, then $f \in \mathcal{S}_k(T_1(N))$.

Proof. Firstly, we observe that the function

$$h(z) = h(x+iy) = \left|f(x+iy)\right|y^{\frac{k}{2}}$$

on $\mathbb{H}$ is invariant under $\Gamma_1(N)$: $h(\gamma z) = h(z)$ for $\gamma \in \Gamma_1(N)$. Let s be a cusp of $\Gamma_1(N)$. Let $\rho \in \mathrm{SL}_2(\mathbb{Z})$ such that $\rho(s) = \infty$. Let $\Gamma_1(N)_s = \{\gamma \in \Gamma_1(N) : \gamma(s) = s\}$ be the stabilizer of s, then

$$\Gamma_1(N)_\infty = \rho\Gamma_1(N)_s\rho^{-1}.$$

Since $z \mapsto z+1 \in \Gamma_1(N)_\infty$, $f[\rho^{-1}]_k$ is invariant under $z \mapsto z+1$. Thus, there exists meromorphic function $\Phi(q)$ in the domain $0 < |q| < r$, with a positive real number r, such that if $q = e^{2\pi iz}$, then

$$f[\rho^{-1}]_k(z) = \Phi(e^{2\pi iz}).$$

Since $f \in \mathcal{S}_k(T_1(N))$, we have

$$\left|f[\rho^{-1}]_k(z)\right| = \left|\Phi(q)\right| = \mathcal{O}(q) = \mathcal{O}(e^{-2\pi y}).$$

It is easy to see that, $h(\rho^{-1}(z)) = \Phi(q)\,\mathrm{Im}(z)^{k/2}$, thus $h(\rho^{-1}(z)) = \mathcal{O}(y^{k/2}e^{-2\pi y})$ and so $h(w) \to 0$ as $w \to s$. Thus h can be viewed as a continuous functions on $\Gamma_1(N)\backslash\mathbb{H}^*$. Since, $\Gamma_1(N)\backslash\mathbb{H}^*$ is compact, $h(z)$ is bounded. Which implies the first assertion. Coversely, if $h(z)$ is bounded, ■

Theorem 17 (Hecke)**.** *If $f \in \mathcal{S}_k(T_1(N))$ has Fourier expansion given by*

$$f(z) = \sum_{n=1}^{\infty} a_n(f)q^n$$

at ∞, then $a_n(f) = \mathcal{O}(n^{\frac{k}{2}})$, where the implicit constant is independent of n.

Proof. There is a constant M such that

$$y^{k/2}\big|f(z)\big| \le M, \quad \forall z \in \mathbb{H}.$$

Now fix y and vary x between 0 and 1. Then $q = e^{2\pi i(x+iy)}$ traverses counterclockwise the circle C_y of radius $e^{-2\pi y}$, centered at zero. By Cauchy's residue theorem,

$$a_f(n) = \frac{1}{2\pi i}\int_{C_y} f(z)q^{-n-1}dq = \int_0^1 f(x+iy)q^{-n}dx.$$

Thus, we obtain

$$|a_f(n)| \le My^{-k/2}e^{-2\pi ny}$$

for any $y > 0$. Choosing $y = 1/n$ gives $a_f(n) = \mathcal{O}(n^{k/2})$. ■

3.1. **Twisted L-functions.**

Definition 6. Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in \mathcal{S}_k(N,\psi)$, let χ be a Dirichlet chracter modulo r, where r is a positive integer relatively prime to N. Define

$$L_f(s,\chi) = \sum_{n=1}^{\infty} \frac{a_n(f)\chi(n)}{n^s}.$$

Let $L_f(s) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}$, then $L_f(s,\chi)$ is called the twist of $L_f(s)$ by χ.

In this section, we consider such twists of $L_f(s)$. Firstly, we recall some elementary facts on the Gauss sum associated to χ. Let us fix a r to be a positive integer, and χ be a primitive Dirichlet character modulo r, then the associated Gauss sum is

$$\tau(\chi) = \sum_{m=1}^{r} \chi(m)e(m/r), \quad e(m/r) = e^{2\pi i(m/r)}.$$

Lemma 18. *The notation being as above, we have:*

(1) $\sum_{m=1}^{r} \chi(m)e(mn/r) = \overline{\chi}(n)\tau($
chi) for every $n \in \mathbb{Z}$.

(2) $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)r$.

(3) $\big|\tau(\chi)\big|^2 = r$.

(4) $\overline{\tau(\chi)} = \chi(-1)\tau(\overline{\chi})$.

Proof. See Lemma 3.63. in [4]. ■

We also recall the definition of the Gamma function, which is denoted by

$$\Gamma(s) = \int_0^{\infty} e^{-x}x^{s-1}dx, \quad (s \in \mathbb{C}).$$

By change of variables: $x \leftrightarrow ax$, we get

$$a^{-s}\Gamma(s) = \int_0^\infty e^{-ax}x^{s-1}dx, \quad (s \in \mathbb{C}, a \in \mathbb{R}, a > 0).$$

3.2. **Analytic continuation and Functional equations.**

Proposition 19. *Let N and r be positive integers, s be a positive divisor of N, and M be the least common multiple of N, r^2, and rs. Let χ (resp. ψ) be a primitve Dirichlet character modulo r (resp. s). Further, let $f = \sum_{n=0}^\infty a_n(f)q^n \in \mathcal{M}_k(N, \psi)$. Then the "twisted series"*

$$f_\chi(z) = \sum_{n=0}^\infty \chi(n)a_n(f)q^n$$

is an element of $\mathcal{M}_k(M, \psi\chi^2)$. Moreover, if f is a cusp form, then so is f_χ.

Proof. Let $\tau(\chi)$ be the Gauss sum

$$\tau(\chi) = \sum_{m=1}^r \psi(m)e(m/r).$$

Also,

$$\tau(\overline{\chi})\chi(n) = \sum_{m=1}^r \sum_{m=1}^r \overline{\chi}(m)e(mn/r).$$

Thus we can write f_χ in terms of f as

$$\tau(\overline{\chi})f_\chi = \sum_{m=1}^r \overline{\chi}(m)f\left[\begin{pmatrix} 1 & \frac{m}{r} \\ 0 & 1 \end{pmatrix}\right]_k.$$

A direct calculation shows that

$$\begin{aligned} \alpha := &\begin{pmatrix} 1 & m/r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -d^2m/r \\ 0 & 1 \end{pmatrix} \\ = &\begin{pmatrix} a + mc/r & b - bcdm/r - cd^2m^2/r^2 \\ c & d - cd^2u/r \end{pmatrix} \end{aligned}$$

so that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$, then $\alpha \in \Gamma_0(M)$ also, because M is the least common multiple of N, r^2, rs, and $M \mid c$, hence $r^2 \mid c$. Also

$$\alpha \equiv \begin{pmatrix} a & \star \\ 0 & d \end{pmatrix} \pmod{s}.$$

Thus,

$$\begin{aligned}\tau(\overline{\chi})f_\chi[\gamma]_k &= \sum_{m=1}^{r}\overline{\chi}(m)f\left[\begin{pmatrix}1 & \frac{m}{r}\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right]\\ &= \sum_{m=1}^{r}\overline{\chi}(m)f\left[\alpha\begin{pmatrix}1 & \frac{d^2m}{r}\\ 0 & 1\end{pmatrix}\right]\\ &= \sum_{m=1}^{r}\overline{\chi}(m)\psi(d')f\left[\begin{pmatrix}1 & \frac{d^2m}{r}\\ 0 & 1\end{pmatrix}\right],\end{aligned}$$

where d' is the lower right entry of α, and $d' \equiv d \pmod{s}$. As $(d,r)=1$, we see that $d^2m \pmod{r}$ forms a complete set of residue classes modulo r as m ranges over all residue classes modulo r. Thus,

$$\begin{aligned}\tau(\overline{\chi})f_\chi[\gamma]_k &= \chi(d)^2\psi(d)\sum_{m=1}^{r}\overline{\chi}(m)f\left[\begin{pmatrix}1 & \frac{m}{r}\\ 0 & 1\end{pmatrix}\right]\\ &= \chi(d)^2\psi(d)\tau(\overline{\chi})f_\chi.\end{aligned}$$

Since $\tau(\overline{\chi}) \neq 0$, we deduce that $f_\chi \in \mathcal{M}_k(M, \psi\chi^2)$ as claimed. ■

For $x \in \mathbb{Z}$, let

$$w_x = \begin{pmatrix}0 & -1\\ x & 0\end{pmatrix}.$$

Lemma 20. *If $f \in \mathcal{S}_k(\Gamma_1(N))$, then $f[w_N]_k \in \mathcal{S}_k(\Gamma_1(N))$. Moreover, if $f \in \mathcal{S}_k(N,\chi)$, then $f[w_N]_k \in \mathcal{S}_k(N,\overline{\chi})$.*

Proof. We first observe that $w_N\Gamma_i(N)w_N^{-1} \subset \Gamma_0(N)$. If $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_1(N)$, then

$$f([w_N]_k)[\gamma]_k = \left(f\left[\begin{pmatrix}d & -c/N\\ -bN & a\end{pmatrix}\right]_k\right)[w_N]_k = f[w_N]_k.$$

Now suppose that $f \in \mathcal{S}_k(N,\chi)$

■

Lemma 21. *Let $(q,r)=1$ and $(u,r)=1$. Let d and w be the integers such that $dr - quw = 1$, and $N = qr^2$. Then*

$$\begin{pmatrix}1 & \frac{u}{r}\\ 0 & 1\end{pmatrix}w_N = rw_q\begin{pmatrix}r & -w\\ -qu & d\end{pmatrix}\begin{pmatrix}1 & \frac{w}{r}\\ 0 & 1\end{pmatrix}.$$

Proof. This is immediate by comparing both sides. ■

Proposition 22. *Keeping the notation from the above proposition, suppose that r is prime to N. Let $g = f[w_N]_k$. Then*

$$f_\chi[w_{r^2N}]_k = w(\chi)g_{\overline{\chi}},$$

where

$$w(\chi) = \psi(r)\chi(N)\tau(\chi)^2/r.$$

Proof. Let m be a positve integer such that $(m, r) = 1$. Then we can find two integrs d and w so that $dr - Nmw = 1$. Then from the above lemma we have:

$$\begin{pmatrix} 1 & \frac{m}{r} \\ 0 & 1 \end{pmatrix} w_{r^2N} = rw_N \begin{pmatrix} r & -w \\ -Nu & d \end{pmatrix} \begin{pmatrix} 1 & \frac{w}{r} \\ 0 & 1 \end{pmatrix}.$$

so that

$$\begin{aligned} f\left[\begin{pmatrix} 1 & \frac{m}{r} \\ 0 & 1 \end{pmatrix} w_{r^2N}\right]_k &= \overline{\psi}(d)g\left[\begin{pmatrix} 1 & \frac{w}{r} \\ 0 & 1 \end{pmatrix}\right]_k \\ &= \psi(r)g\left[\begin{pmatrix} 1 & \frac{w}{r} \\ 0 & 1 \end{pmatrix}\right]_k. \end{aligned}$$

We also have

$$\begin{aligned} \tau(\overline{\chi})f_\chi[w_{r^2N}]_k &= \sum_{m=1}^{r} \overline{\chi}(m) f\left[\begin{pmatrix} 1 & \frac{m}{r} \\ 0 & 1 \end{pmatrix} w_{r^2N}\right]_k \\ &= \sum_{m=1}^{r} \overline{\chi}(m)\psi(r)g\left[\begin{pmatrix} 1 & \frac{w}{r} \\ 0 & 1 \end{pmatrix}\right]_k \end{aligned}$$

As m ranges over coprime residue classes modulo r, so does w. Since $Nmw \equiv -1 \pmod r$, we have $\overline{\chi}(Nmw) = \chi(-1))$ so that $\overline{\chi}(m) = \chi(-N)\chi(w)$. Thus

$$\begin{aligned} \tau(\overline{\chi})f_\chi[w_{r^2N}]_k &= \chi(-N)\psi(r)\sum_{w=1}^{r} \overline{\chi}(w) g\left[\begin{pmatrix} 1 & \frac{w}{r} \\ 0 & 1 \end{pmatrix}\right]_k \\ &= \chi(-N)\psi(r)\tau(\chi)g_{\overline{\chi}}. \end{aligned}$$

From Lemma 18(4) we have that

$$\overline{\tau(\psi)} = \psi(-1)\tau(\overline{\psi}).$$

We obtain

$$\overline{\tau(\chi)}f_\chi[w_{r^2N}]_k = \chi(N)\psi(r)\tau(\chi)g_{\overline{\chi}}$$

so that

$$rf_\chi[w_{r^2N}]_k = \chi(N)\psi(r)\tau(\chi)^2 g_{\overline{\chi}}$$

which is the desired result. ■

Let ψ be an arbitrary Dirichlet character modulo N, given

$$f = \sum_{n=1}^{\infty} a_n(f) q^n \in \mathcal{S}_k(N, \psi)$$

and a primitive Dirichlet character χ modulo r, where r is coprime to N, then we can associate the "twisted" L-series to f:

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{a_n(f)\chi(n)}{n^s}$$

where $a_n(f)$ are the Fourier coefficients of f at ∞. Furthermore, define the **completed** twisted L-function of f as:

$$\Lambda_f(s, \chi) = \left(\frac{\sqrt{r^2 N}}{2\pi} \right)^s \Gamma(s) L_f(s, \chi).$$

Then we have the following functional equation for $L_f(s, \chi)$

Theorem 23. *Keeping the notation as above and $f \in \mathcal{S}_k((N), \psi)$. Then $L_f(s, \chi)$ is absolutely convergent for $\mathrm{Re}(s) > 1 + (k/2)$, and can be holomorphicaly continued to the whole complex plane. Furthermore, $\Lambda_f(s, \chi)$ is an entire function, bounded in vertical strips and satisfying the functional equation*

$$\Lambda_f(s, \chi) = i^k w(\chi) \Lambda_g(k - s, \overline{\chi}),$$

where

$$w(\chi) = \psi(r)\chi(N)\tau(\chi)^2/r$$

and $g = f[w_N]_k$, with $w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

Proof. The absolute convergence for $\mathrm{Re}(s) > k/2 + 1$ follows from Theorem 17. Now suppose that the second assertion of theorem is true for $r = 1$ and $\chi = 1$. Then from propsitions 19 and 22 we have that $f_\chi \in \mathcal{S}_k(r^2 N, \psi\chi^2)$ and

$$\Lambda_{f_\chi}(s, 1) = \Lambda_f(s, \chi),$$
$$\Lambda_{f_\chi[w_{r^2N}]}(k - s, 1) = \psi(r)\chi(N)\tau(\chi)^2 r^{-1} \Lambda_g(k - s, \overline{\chi}).$$

Furthermore, from the case of $r = 1$ and $\chi = 1$, we have the following

$$\Lambda_{f_\chi}(s, 1) = i^k \Lambda_{f_\chi[w_{r^2N}]}(k - s, 1).$$

Thus

$$\Lambda_f(s, \chi) = i^k \psi(r)\chi(N)\tau(\chi)^2 r^{-1} \Lambda_g(k - s, \overline{\chi}) = i^k w(\chi) \Lambda_g(k - s, \overline{\chi}).$$

It remains to prove the theorem for $r = 1$ and $\chi = 1$. We need the following lemma. ∎

We can view the completed L-function $\Lambda_f(s,1)$ as **Mellin** transform of f in the following sense.

Lemma 24 (Mellin Transformation). *Keeping the same notations. For* $\mathrm{Re}(s) > k/2+1$ *we have the following equalities:*

$$\Lambda_f(s,1) = N^{\frac{s}{2}} \int_0^\infty f(iy) y^{s-1} dy$$
$$\Lambda_g(s,1) = N^{\frac{s}{2}} \int_0^\infty g(iy) y^{s-1} dy,$$

where $g = f[w_N]_k$.

Proof. It suffices to show the first equality. Formally we have

$$\int_0^\infty f(iy) y^{s-1} dy = \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi n y} y^{s-1} dy = (2\pi)^{-s}\Gamma(s) L_f(s,1). \tag{7}$$

We need to check the validity of this equation by checking the convergence on both sides. For any $\varepsilon > 0$ (small enough) then from Lemma 16 we have

$$\left| \int_0^\varepsilon f(iy) y^{s-1} dy \right| \leq M \int_0^\varepsilon y^{-k/2} y^{k/2} dy \to 0 \quad (\varepsilon \to 0)$$

if $\mathrm{Re}(s) > k/2+1$.

Further for any $E > 0$ we have

$$\left| \int_E^\infty f(iy) y^{s-1} dy \right| \leq M' \int_E^\infty e^{-2\pi y} y^{\mathrm{Re}(s)-1} dy \to 0 \quad (E \to \infty)$$

for any $s \in \mathbb{C}$.

For the remaining part of the integral we have

$$\int_\varepsilon^E f(iy) y^{s-1} dy = \sum_{n=1}^\infty a_n \int_\varepsilon^E e^{-2\pi n y} y^{s-1} dy$$

since $\sum_{n=1}^\infty a_n e^{-2\pi n y}$ is uniformy convergent for $y \geq \varepsilon$. Now for any $\delta > 0$ we choose M large enough so that the following is satisified

$$\begin{aligned} \left| \sum_{n>M} a_n \int_\varepsilon^E e^{-2\pi n y} y^{s-1} dy \right| &\leq \sum_{n>M} |a_n| \int_\varepsilon^E e^{-2\pi n y} y^{\mathrm{Re}(s)-1} dy \\ &= \Gamma(\sigma)(2\pi)^{-\mathrm{Re}(s)} \sum_{n>M} |a_n| n^{-\mathrm{Re}(s)} < \delta. \end{aligned}$$

From which it follows that

$$\left|\int_0^\infty f(iy)y^{s-1}dy - \sum_{n=1}^{M} a_n \int_0^\infty e^{-2\pi n y} y^{s-1}dy\right|$$

$$= \lim_{\substack{\varepsilon\to 0\\ E\to\infty}} \left|\int_\varepsilon^E f(iy)y^{s-1}dy - \sum_{n=1}^{M} a_n \int_\varepsilon^E e^{-2\pi n y} y^{s-1}dy\right| \leq \delta.$$

This shows that equation 7 is valid for $\mathrm{Re}(s) > k/2+1$. Same reasoning gives the second equality. ■

Proof of Theorem 23. We continue the proof. Let $A = N^{-\frac{1}{2}}$. Then

$$\int_0^\infty f(iy)y^{s-1}dy = \int_0^A f(iy)y^{s-1}dy + \int_A^\infty f(iy)y^{s-1}dy$$

where the first integral converges for $\mathrm{Re}(s) > k/2+1$, and the second integral converges for every s. By change of variables $y \leftrightarrow 1/ny$, we obtain

$$\int_0^A f(iy)y^{s-1}dy = \int_A^\infty f(i/Ny)N^{-s}y^{-s-1}dy = i^k N^{k/2-s}\int_A^\infty g(iy)y^{k-1-s}dy$$

the last equality is because $f(i/Ny) = N^{k/2}(iy)^k g(iy)$, and the last integral is convergent for any s. Similarly

$$\int_A^\infty f(iy)y^{s-1}dy = i^k N^{k/2-s}\int_0^A g(iy)y^{k-1-s}dy \quad (\mathrm{Re}(s) > k/2+1).$$

Therefore, we see that

$$\Lambda_f(s,1) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L_f(s,1) = N^{\frac{s}{2}}\int_0^\infty f(iy)y^{s-1}dy$$

can be holomorphically continued to the whole complex plane and we have the following functional equation

$$\Lambda_f(s,1) = i^k\Lambda_g(k-s,1)$$

which proves the case $r=1$ and $\chi=1$ and thus completes the proof of the theorem. ■

The arguments presented above indicates a close connection between Dirichlet series with a functional equation and modular forms. In particular, we obtained Dirichlet series with a functional equation from cusp forms. So the questiona naturally arises: Can one go the other way? Does every Dirichlet series with the right type of functional equation come from some modular form, i.e., is it of the form $L_f(s,\chi)$ for some modular form $f \in (M)_k(N,\chi)$ and Dirichlet character χ modulo N. Hecke [1936] and Weil [1967] shoed that the answer to these questions is yes, but with some qualifications, for more details we refer to [3].

References

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